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LETTER TO THE EDITOR

On the Pauli–Van Vleck formula for arbitrary quadratic systems with memory in one dimension

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Abstract. The propagator for an *arbitrary* quadratic system with memory in one dimension is calculated using the Schwinger action principle. The propagator has the Pauli–Van Vleck form.

In order to motivate this work and our method of calculation we would first like to make a brief comment on some results published earlier concerning the calculation of the propagator of quadratic systems with memory (Papadopoulos 1974, Maheshwari 1975, Khandekar *et al* 1983). The most general system considered so far is described by the action (Khandekar *et al* 1983)

$$S=\int_0^T\left(\tfrac{1}{2}m\dot{x}^2-\int_0^T G(t,s)(x(t)-x(s))^2\,\mathrm{d}s\right)\mathrm{d}t \tag{1}$$

where $G(t,s)$ is an arbitrary symmetrical function of its arguments. This action includes, as a particular case, the Bezak kernel (Bezak 1970) whose propagator was obtained by Papadopoulos (1974) and Maheshwari (1975). Using path integral techniques Papadopoulos, Maheswari and Khandekar *et al* calculate the corresponding propagator and write it in terms of a quasi Pauli–Van Vleck formula:

$$\langle x'',T|x',0\rangle=C_f\left(\frac{1}{2\pi\mathrm{i}\hbar}\left|\frac{\partial^2 S}{\partial x'\partial x''}\right|\right)^{1/2}\exp\left(\frac{\mathrm{i}}{\hbar}S\right) \tag{2}$$

where S is the classical action and $C_f(T)$ is an extra normalisation factor such that $|C_f(T)|^2\neq 1$. In the Bezak case $C_f=(\Omega T/\sin\Omega T)^{1/2}$ and in the more general situation of the action (1), C_f is given in terms of the solutions of an integrodifferential equation together with an oscillator-type differential equation with time-dependent frequency. Of course, it is still necessary to solve the classical integrodifferential equation of motion in order to evaluate the classical action.

Without going into any details of the path integral calculation it is very easy to convince oneself that the result (2) is wrong. In fact, the composition property (unitarity)

$$\int_{-\infty}^{+\infty}\langle x''',t'|x'',t''\rangle\langle x'',t''|x',t'\rangle\,\mathrm{d}x''=\langle x''',t'|x',t'\rangle=\delta(x'''-x') \tag{3}$$

would be satisfied by propagators of the form (2) only if

$$|C_f|^2 = 1 \tag{4}$$

for any classical action given by

$$S = A(T)x'^2 + B(T)x''^2 + C(T)x'x''. \tag{5}$$

Incidentally, this expression for S is more general than that corresponding to (1). The immediate calculation of the left-hand side of (3) using (2), together with $\langle x', t'|x'', t''\rangle = \langle x'', t''|x', t'\rangle^*$, gives

$$|C_f|^2 \exp\left(\frac{\mathrm{i}A}{\hbar}(x'^2 - x'''^2)\right)\delta(x' - x''') \tag{6}$$

from a which condition (4) follows.

Confronted with such an state of affairs it became apparent that a completely independent calculation of such propagators was necessary. Using the well known Schwinger action principle for quantum mechanics we have been able to extend all previous calculations, obtaining the propagator for an *arbitrary* one-dimensional system with memory described by the first order operator action ($m = 1$)

$$S = \int_0^T L\,\mathrm{d}t = \int_0^T \left(p\dot{x} - \tfrac{1}{2}p^2 - \int_0^T G(t, s;\, T)\{x(t), x(s)\,\mathrm{d}s\right)\mathrm{d}t. \tag{7}$$

Here x and p are the usual position and momentum operators and $G(t, s, T)$ is an arbitrary symmetrical function of t, s. The bracket $\{x(t), x(s)\}$ stands for the anticommutator of the corresponding position operators, which is necessary in order to have a Hermitian Lagrangian operator up to a total time derivative. We emphasise that the action (1) discussed by Khandekar *et al* (1983), together with those corresponding to all systems having interactions which are local in time, are just particular cases of (7) corresponding to different choices of the kernel G.

Our result for the propagator related to the action (7) is

$$\langle x'', T|x', 0\rangle = \left(\frac{|\dot{\eta}(0)|}{2\pi\mathrm{i}\hbar}\right)^{1/2} \exp\frac{\mathrm{i}}{2\hbar}(\dot{\eta}(T)x''^2 - \dot{\xi}(0)x'^2 - 2\dot{\eta}(0)x''x') \tag{8}$$

given *only* in terms of the functions $\eta(t)$ and $\xi(t)$ which are solutions of the classical equation of motion:

$$\ddot{z}(t) + \int_0^T G(t, s)z(s)\,\mathrm{d}s = 0. \tag{9}$$

Such functions satisfy the following boundary conditions

$$\begin{aligned} &\xi(0) = 1 \qquad && \eta(0) = 0 \\ &\xi(T) = 0 \qquad && \eta(T) = 1. \end{aligned} \tag{10}$$

The notation in (8) is $\dot{z}(\tau) = \mathrm{d}z(t)/\mathrm{d}t|_{t=\tau}$. It can also be shown that the function S appearing in the exponential term of (8), $\exp(\mathrm{i}S/\hbar)$, is indeed the classical action.

The calculation of (8) proceeds essentially along the same lines as in the trivial case of the harmonic oscillator with constant frequency. We regard this as a virtue of the method instead of a drawback. The only point that requires special care is the verification for the general case (7) of some integrability conditions which arise from the expression for $\delta\langle x'', T|x', 0\rangle$ which is provided by the action principle. The details of our calculation will be presented elsewhere.

Finally we comment that our propagator (8) exhausts all possible cases of quadratic systems without external forces in one dimension and that it is obviously of the Pauli–Van Vleck form. We conjecture that this result generalises to the three-dimensional case for quadratic interactions.

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